

Schrödinger geometries arising from Yang-Baxter deformations

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Abstract

We present further examples of the correspondence between solutions of type IIB supergravity and classical r -matrices satisfying the classical Yang-Baxter equation (CYBE). In the previous works, classical r -matrices have been composed of generators of only one of either $\mathfrak{so}(2,4)$ or $\mathfrak{so}(6)$. In this paper, we consider some examples of r -matrices with both of them. The r -matrices of this kind contain (generalized) Schrödinger spacetimes and gravity duals of dipole theories. It is known that the generalized Schrödinger spacetimes can also be obtained via a certain class of TsT transformations called null Melvin twists. The metric and NS-NS two-form are reproduced by following the Yang-Baxter sigma-model description.

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1 Introduction

The gauge/gravity correspondence provides a fascinating arena for studying new aspects of string theory. A well-studied example is the duality between type IIB superstring theory on $\text{AdS}_5 \times \text{S}^5$ and the $\mathcal{N}=4$ super Yang-Mills (SYM) theory in four dimensions [1]. This duality enjoys a powerful property, integrability (For a big review, see [2]). It enables one to compute some physical quantities exactly even at finite coupling without supersymmetries. Thus the integrable structure behind the duality will be more and more important in the future study.

The integrable structure may provide another insight as well. It is integrable deformations of the AdS/CFT correspondence. It is well recognized that type IIB string theory on $\text{AdS}_5 \times \text{S}^5$ [3], which is often called the $\text{AdS}_5 \times \text{S}^5$ superstring, is classically integrable [4].

A systematic way to study the deformations is to employ the Yang-Baxter sigma-model formulation, which was originally proposed by Klimcik [5] for principal chiral models¹ and generalized by Delduc-Magro-Vicedo [9] to coset sigma models. In this formalism, a deformed target space is determined by specifying the associated classical r -matrix satisfying the modified classical Yang-Baxter equation (mCYBE). This formalism has been extended to the case with the Wess-Zumino-Witten (WZW)-like term [10]². One may consider a generalization of it to the standard classical Yang-Baxter equation (CYBE) [13]. A three-dimensional Schrödinger spacetime is contained here as a simple case with the rank 1 and for this case the classical integrable structure has been recognized as Jordanian twists [14]³.

The formalism of the Yang-Baxter deformations is applicable to the $\text{AdS}_5 \times S^5$ superstring [17]. Then it has been generalized to the CYBE case [18]. In comparison to the mCYBE, there are two advantages. The first is that partial deformations of $\text{AdS}_5 \times S^5$ are possible. For example, one may consider a deformation of either AdS_5 or S^5 , while it seems likely, so far as we see the current achievement, that both of them are inevitably deformed in the case of mCYBE [17]. The second is that there are a variety of solutions of the CYBE. In fact, we have already found various examples of classical r -matrices which correspond to solutions of type IIB supergravity. For example, the Lunin-Maldacena-Frolov background [19, 20] and the gravity duals of non-commutative gauge theories [21, 22] have been reproduced in [23] and [24], respectively, by following the Yang-Baxter sigma-model description. Thus, the correspondence of this kind may be called the gravity/CYBE correspondence [23] (For a short summary see, [25]).

A remarkable point of the gravity/CYBE correspondence is the connection to TsT transformations [26, 27]. That is, a certain class of classical r -matrices satisfying the CYBE is related to a solution generation techniques in type IIB supergravity. More strikingly, it is shown in [28] that this connection holds even for Yang-Baxter deformations of a non-integrable background, $\text{AdS}_5 \times T^{1,1}$ [29]. Motivated by these developments, our purpose here is to pursue the relation to TsT transformations and discover further examples of the gravity/CYBE correspondence.

Indeed, we will present further examples of the gravity/CYBE correspondence. In the previous works, classical r -matrices have been composed of generators of only one of either $\mathfrak{so}(2, 4)$ or $\mathfrak{so}(6)$. In the preset paper, we consider some examples of r -matrices with both

¹ For the $SU(2)$ case, a q -deformed algebra and its affine extension have been revealed in [6–8].

² For the related progress on integrability with the WZW-term, see also [11, 12].

³ It may be significant to unveil the relation between the classical r -matrix and the general invariant two-form in the coset construction [15]. It is also interesting to figure out the relation to the construction [16].

of them, which are schematically of the form;

$$r = \sum_i (a_i \otimes b_i - b_i \otimes a_i) \quad \text{with} \quad a_i \in \mathfrak{so}(2, 4), \quad b_i \in \mathfrak{so}(6). \quad (1.1)$$

The classical r -matrices of this kind contain (generalized) Schrödinger spacetimes and gravity duals of dipole theories. It is known that the generalized Schrödinger spacetimes can also be obtained via a certain class of TsT transformations called null Melvin twists as discussed in [30] and [31, 32]. The metric and NS-NS two-form are reproduced by following the Yang-Baxter sigma-model description.

This paper is organized as follows. Section 2 gives a short review of Yang-Baxter deformations of the $\text{AdS}_5 \times \text{S}^5$ superstring based on the CYBE. Section 3 presents classical r -matrices associated with (generalized) Schrödinger spacetimes. Section 4 considers dipole deformations of the $\text{AdS}_5 \times \text{S}^5$ background with the corresponding r -matrices. Section 5 is devoted to conclusion and discussion. In Appendix A, we summarize our convention and notation of the $\mathfrak{so}(2, 4)$ and $\mathfrak{so}(6)$ generators. In Appendix B, the T-duality rules are listed. Appendix C gives a derivation of three-parameter dipole deformations of $\text{AdS}_5 \times \text{S}^5$.

2 Yang-Baxter deformations of $\text{AdS}_5 \times \text{S}^5$

Let us first recall the formulation of Yang-Baxter sigma models.

The Yang-Baxter sigma-model description was originally developed for purely bosonic non-linear sigma models [5, 9]. It is now generalized to supersymmetric cases, and integrable deformations of the $\text{AdS}_5 \times \text{S}^5$ superstring can be described based on the mCYBE [17] and the CYBE [18].

Here we consider the latter case [18], where the deformed classical action is given by

$$S = -\frac{1}{4}(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \text{STr} \left(A_\alpha d \circ \frac{1}{1 - \eta R_g \circ d} A_\beta \right), \quad (2.1)$$

where the left-invariant one-form is defined as

$$A_\alpha \equiv g^{-1} \partial_\alpha g, \quad g \in SU(2, 2|4). \quad (2.2)$$

When $\eta = 0$, the classical action (2.1) is reduced to the undeformed one [3]. Here the string world-sheet metric is taken to be flat i.e., $\gamma^{\alpha\beta} = \text{diag}(-1, 1)$. The anti-symmetric tensor $\epsilon^{\alpha\beta}$ is normalized as $\epsilon^{\tau\sigma} = 1$.

The operator R_g is defined as

$$R_g(X) \equiv g^{-1}R(gXg^{-1})g, \quad (2.3)$$

where a linear operator R is a solution of CYBE rather than mCYBE. The R-operator is related to a classical r -matrix in the tensorial notation through

$$R(X) = \text{STr}_2[r(1 \otimes X)] = \sum_i (a_i \text{STr}(b_i X) - b_i \text{STr}(a_i X)) \quad (2.4)$$

with $r = \sum_i a_i \wedge b_i \equiv \sum_i (a_i \otimes b_i - b_i \otimes a_i).$

The generators a_i, b_i are some elements of $\mathfrak{su}(2, 2|4)$. The supertrace STr of $(4|4) \times (4|4)$ supermatrix is defined as

$$\text{STr} \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = \text{Tr}(A) - \text{Tr}(D), \quad (2.5)$$

where each of the blocks A, B, C, D is a 4×4 matrix of complex numbers, which plays a crucial role in our argument.

Note that $\mathfrak{su}(2, 2|4)$ enjoys the \mathbb{Z}_4 -grading property and one can introduce the projectors P_k ($k = 0, 1, 2, 3$) from $\mathfrak{su}(2, 2|4)$ to its \mathbb{Z}_4 -graded components $\mathfrak{su}(2, 2|4)^{(k)}$. In particular, $\mathfrak{su}(2, 2|4)^{(0)}$ is a gauge symmetry, $\mathfrak{so}(1, 4) \oplus \mathfrak{so}(5)$. Then the operator d is defined as a linear combination of P_k ,

$$d \equiv P_1 + 2P_2 - P_3. \quad (2.6)$$

The numerical coefficients are fixed by requiring the kappa-symmetry [18].

To evaluate the action, it is convenient to rewrite the metric part and NS-NS two-form coupled part of the Lagrangian (2.1) into the following form,

$$\begin{aligned} L_G &= \frac{1}{2} \text{STr} [A_\tau P_2(J_\tau) - A_\sigma P_2(J_\sigma)] , \\ L_B &= \frac{1}{2} \text{STr} [A_\tau P_2(J_\sigma) - A_\sigma P_2(J_\tau)] , \end{aligned} \quad (2.7)$$

where J_α is a projected current defined as

$$J_\alpha \equiv \frac{1}{1 - 2\eta R_g \circ P_2} A_\alpha. \quad (2.8)$$

One can read off the deformed metric and NS-NS two-form from L_G and L_B , respectively. It is quite messy but straightforward. It would be an easy exercise to derive the metric and NS-NS two-form by following the previous examples [23–27].

3 Integrability of Schrödinger spacetimes

In this section, we will deduce Schrödinger spacetimes from classical r -matrices depending on both $\mathfrak{so}(2,4)$ and $\mathfrak{so}(6)$. Consequently, the classical integrability of the spacetimes automatically follows from the Yang-Baxter sigma model formulation. After presenting the well-known Schrödinger spacetime in subsection 3.1, we consider three-parameter generalizations in subsection 3.2. More complicated examples are presented in subsection 3.3.

3.1 Schrödinger geometries as Yang-Baxter deformations

Inspired by the spirit of the gravity/CYBE correspondence, we have found that the following classical r -matrix

$$r = -\frac{i\beta}{4\eta} p_- \wedge (h_4 + h_5 + h_6) \quad (3.1)$$

corresponds to the Schrödinger background. Here p_- is the light-cone generator in $\mathfrak{so}(2,4)$ and h_4, h_5 and h_6 are the Cartan generators in $\mathfrak{so}(6)$. Hence the r -matrix in (3.1) is indeed of the form (1.1). The parameter β measures the deformation⁴. For the details of our convention and notation of the generators, see Appendix A.

To find the metric and NS-NS two-form from (2.7), we need to evaluate the projected deformed current $P_2(J_\alpha)$. By solving the equation

$$(1 - 2\eta P_2 \circ R_g) P_2(J_\alpha) = P_2(A_\alpha), \quad (3.2)$$

which is obtained from the definition (2.8), the deformed current is evaluated as

$$\begin{aligned} P_2(J_\alpha) = & c^1 \gamma_1^a + c^2 \gamma_2^a + c^3 \gamma_3^a + c^0 \gamma_0^a + c^5 \gamma_5^a \\ & + d^1 \gamma_1^s + d^2 \gamma_2^s + d^3 \gamma_3^s + d^4 \gamma_4^s + d^5 \gamma_5^s, \end{aligned} \quad (3.3)$$

with the coefficients⁵

$$\begin{aligned} c^1 &= \frac{\partial_\alpha x^1}{2z}, & c^2 &= \frac{\partial_\alpha x^2}{2z}, & c^5 &= \frac{\partial_\alpha z}{2z}, \\ c^3 &= -\frac{\beta^2 \partial_\alpha x^+}{2\sqrt{2}z^3} + \frac{1}{2\sqrt{2}z} \left(\partial_\alpha x^+ - \partial_\alpha x^- + \beta \left(\partial_\alpha \chi + \frac{1}{2} \sin^2 \mu (\partial_\alpha \psi + \cos \theta \partial_\alpha \phi) \right) \right), \\ c^0 &= +\frac{\beta^2 \partial_\alpha x^+}{2\sqrt{2}z^3} + \frac{1}{2\sqrt{2}z} \left(\partial_\alpha x^+ + \partial_\alpha x^- - \beta \left(\partial_\alpha \chi + \frac{1}{2} \sin^2 \mu (\partial_\alpha \psi + \cos \theta \partial_\alpha \phi) \right) \right), \\ d^1 &= -\frac{i}{2} \partial_\alpha \mu, & d^3 &= -\frac{i}{4} \sin \mu \partial_\alpha \theta, & d^5 &= \frac{i}{2z^2} \cos \mu (z^2 \partial_\alpha \chi - \beta \partial_\alpha x^+), \end{aligned}$$

⁴ The parameter η is not an essential deformation parameter because it is canceled in (2.8).

⁵For the conventions of the gamma-matrices, see Appendix A.

$$\begin{aligned}
d^2 &= \frac{i}{4z^2} \cos \frac{\theta}{2} \sin \mu (2\beta \partial_\alpha x^+ - z^2 (\partial_\alpha \phi + 2\partial_\alpha \chi + \partial_\alpha \psi)) , \\
d^4 &= \frac{i}{4z^2} \sin \frac{\theta}{2} \sin \mu (2\beta \partial_\alpha x^+ + z^2 (\partial_\alpha \phi - 2\partial_\alpha \chi - \partial_\alpha \psi)) .
\end{aligned} \tag{3.4}$$

Plugging the above expression of $P_2(J_\alpha)$ with (2.7), the resulting metric and NS-NS two-form turn out to be

$$\begin{aligned}
ds^2 &= \frac{-2dx^+ dx^- + (dx^1)^2 + (dx^2)^2 + dz^2}{z^2} - \beta^2 \frac{(dx^+)^2}{z^4} + ds_{\mathbb{S}^5}^2 , \\
B_2 &= \frac{\beta}{z^2} dx^+ \wedge (d\chi + \omega) .
\end{aligned} \tag{3.5}$$

Here the line element $ds_{\mathbb{S}^5}^2$ is measured by the metric of \mathbb{S}^5 with the coordinates $(\chi, \mu, \psi, \theta, \phi)$,

$$\begin{aligned}
ds_{\mathbb{S}^5}^2 &= (d\chi + \omega)^2 + ds_{\mathbb{CP}^2}^2 , \\
ds_{\mathbb{CP}^2}^2 &= d\mu^2 + \sin^2 \mu (\Sigma_1^2 + \Sigma_2^2 + \cos^2 \mu \Sigma_3^2) .
\end{aligned} \tag{3.6}$$

Namely, the round \mathbb{S}^5 is expressed as an \mathbb{S}^1 -fibration over \mathbb{CP}^2 , where χ is the fiber coordinate and ω is a one-form potential of the Kähler form on \mathbb{CP}^2 . The symbols Σ_i ($i = 1, 2, 3$) and ω are defined as

$$\begin{aligned}
\Sigma_1 &= \frac{1}{2}(\cos \psi d\theta + \sin \psi \sin \theta d\phi) , \\
\Sigma_2 &= \frac{1}{2}(\sin \psi d\theta - \cos \psi \sin \theta d\phi) , \\
\Sigma_3 &= \frac{1}{2}(d\psi + \cos \theta d\phi) , \quad \omega = \sin^2 \mu \Sigma_3 .
\end{aligned} \tag{3.7}$$

The metric of the deformed AdS_5 part is nothing but the metric that was originally proposed in [33, 34]. This metric preserves a non-relativistic conformal symmetry called the Schrödinger symmetry [35]. This metric can be reproduced via a coset construction [15].

As a result, we have proven the classical integrability of the string sigma model whose target space is given by the Schrödinger background (3.5) in the sense that the Lax pair has been constructed⁶.

3.2 Three-parameter generalizations

In the previous subsection, we have considered a one-parameter deformation. Then, one may consider multi-parameter deformations. An example of three-parameter deformation is

⁶ Note here that non-integrability of various non-relativistic backgrounds was argued in [36], but the Schrödinger spacetime with the dynamical critical exponent $z = 2$ has not been covered. Hence, our result is not in contradiction with [36].

described by the following classical r -matrix,

$$r(\vec{\beta}) = -\frac{i}{4\eta} p_- \wedge (\beta_1 h_4 + \beta_2 h_5 + \beta_3 h_6), \quad (3.8)$$

where $\vec{\beta} \equiv (\beta_1, \beta_2, \beta_3)$ are the real deformation parameters.

Since the calculations to find the metric and NS-NS two-form are completely parallel to the previous section, we do not repeat them here. With the S^5 metric (A.17), the final expressions are written as

$$\begin{aligned} ds^2 &= \frac{-2dx^+ dx^- + (dx^1)^2 + (dx^2)^2 + dz^2}{z^2} - \frac{f(\vec{\beta})(dx^+)^2}{4z^4} + ds_{S^5}^2, \\ B_2 &= \frac{1}{4z^2} dx^+ \wedge K(\vec{\beta}), \end{aligned} \quad (3.9)$$

where $f(\vec{\beta})$ and $K(\vec{\beta})$ are a scalar function and a one-form on S^5 , respectively, depending on $\vec{\beta}$. They are explicitly defined as

$$\begin{aligned} f(\vec{\beta}) &= \beta_1^2 + \beta_2^2 + 2\beta_3^2 + 2(\beta_1^2 - \beta_2^2) \cos \theta \sin^2 \mu - (\beta_1^2 + \beta_2^2 - 2\beta_3^2) \cos 2\mu, \\ K(\vec{\beta}) &= (2(\beta_1 - \beta_2) \cos \theta \sin^2 \mu - (\beta_1 + \beta_2 - 2\beta_3) \cos 2\mu + \beta_1 + \beta_2 + 2\beta_3) d\chi \\ &\quad + (\beta_1 + \beta_2 + (\beta_1 - \beta_2) \cos \theta) \sin^2 \mu d\psi \\ &\quad + (\beta_1 - \beta_2 + (\beta_1 + \beta_2) \cos \theta) \sin^2 \mu d\phi. \end{aligned} \quad (3.10)$$

The above expressions are quite messy and it is convenient to rewrite them in terms of another coordinate system of S^5 . In fact, by following the argument in Appendix A, one can rewrite them into the following simpler form:

$$\begin{aligned} ds^2 &= \frac{-2dx^+ dx^- + (dx^1)^2 + (dx^2)^2 + dz^2}{z^2} - \left(\sum_{i=1}^3 \beta_i^2 \mu_i^2 \right) \frac{(dx^+)^2}{z^4} + ds_{S^5}^2, \\ B_2 &= \frac{1}{z^2} dx^+ \wedge \left(\sum_{i=1}^3 \beta_i \mu_i^2 d\psi_i \right), \quad \mu_1^2 + \mu_2^2 + \mu_3^2 = 1. \end{aligned} \quad (3.11)$$

Here μ_1 , μ_2 and μ_3 are the S^5 coordinates defined as

$$\mu_1 = \cos \zeta \sin r, \quad \mu_2 = \sin \zeta \sin r, \quad \mu_3 = \cos r. \quad (3.12)$$

This background (3.11) nicely agrees with the one obtained by Bobev and Kundu [31] via null Melvin twists of $\text{AdS}_5 \times S^5$. As a result, this background (3.11) also gives rise to an integrable background of type IIB string theory. A remarkable point is that the deformation term of the AdS_5 part depends on the S^5 coordinates. This dependence does not break the

Schrödinger symmetry. It can be generalized beyond the TsT transformations. For example, the dependence of this type has been studied in [37]. It would be interesting to study whether the classical integrability is preserved for the deformation argued in [37]. The stability of the solution might be related to the classification of possible classical r -matrices.

Note that when the deformation parameters take the special values,

$$\beta_1 = \beta_2 = \beta_3 = \beta, \quad (3.13)$$

the background (3.11) reduces to the Schrödinger geometry (3.5) by taking account of the relation,

$$d\chi + \omega = \mu_1^2 d\psi_1 + \mu_2^2 d\psi_2 + \mu_3^2 d\psi_3. \quad (3.14)$$

Finally it is worth noting that the Schrödinger spacetimes discussed so far are non-supersymmetric for generic values. This is basically because T-dualities are taken for R-symmetry directions and then all of the supersymmetries are broken. For supersymmetric Schrödinger backgrounds, see the following subsection.

3.3 Other examples

It would be worth giving more examples. We will give five examples of classical r -matrices satisfying the CYBE. The r -matrices always contain p_- from $\mathfrak{so}(2, 4)$, and the other generators are picked up from $\mathfrak{so}(6)$, which are not the Cartan generators.

To present the examples, the following polar coordinates of S^5 are more convenient:

$$\begin{aligned} n_1 &= \mu_1 \cos \psi_1, & n_2 &= \mu_1 \sin \psi_1, \\ n_3 &= \mu_2 \cos \psi_2, & n_4 &= \mu_2 \sin \psi_2, \\ n_5 &= \mu_3 \cos \psi_3, & n_6 &= \mu_3 \sin \psi_3, \end{aligned} \quad (3.15)$$

where $\mu_1 = \cos \zeta \sin r$, $\mu_2 = \sin \zeta \sin r$, $\mu_3 = \cos r$.

The above coordinates satisfy the following condition:

$$(n_1^2 + n_2^2) + (n_3^2 + n_4^2) + (n_5^2 + n_6^2) = \mu_1^2 + \mu_2^2 + \mu_3^2 = 1. \quad (3.16)$$

1) classical r -matrix with γ_1^s

The first examples is a classical r -matrix composed of p_- and γ_1^s like

$$r = -\frac{i}{4} p_- \wedge \gamma_1^s. \quad (3.17)$$

This r -matrix gives rise to the following metric and NS-NS two-form:

$$\begin{aligned}
ds^2 &= ds_{\text{AdS}_5}^2 + ds_{\text{S}^5}^2 + \eta^2(\mu_1^2 \cos^2 \psi_1 + \mu_3^2 \cos^2 \psi_3) \frac{(dx^+)^2}{z^4} \\
&= ds_{\text{AdS}_5}^2 + ds_{\text{S}^5}^2 + \eta^2(n_1^2 + n_5^2) \frac{(dx^+)^2}{z^4}, \\
B_2 &= \frac{\eta}{z^2} dx^+ \wedge ((\mu_1 \cos \psi_1) d(\mu_3 \cos \psi_3) - (\mu_3 \cos \psi_3) d(\mu_1 \cos \psi_1)) \\
&= \frac{\eta}{z^2} dx^+ \wedge (n_1 dn_5 - n_5 dn_1).
\end{aligned} \tag{3.18}$$

2) classical r -matrix with n_{13}

The second example is a classical r -matrix,

$$r = \frac{1}{2} p_- \wedge n_{13}. \tag{3.19}$$

The resulting metric and NS-NS two-form are given by

$$\begin{aligned}
ds^2 &= ds_{\text{AdS}_5}^2 + ds_{\text{S}^5}^2 - \eta^2(\mu_1^2 \cos^2 \psi_1 + \mu_2^2 \cos^2 \psi_2) \frac{(dx^+)^2}{z^4} \\
&= ds_{\text{AdS}_5}^2 + ds_{\text{S}^5}^2 - \eta^2(n_1^2 + n_3^2) \frac{(dx^+)^2}{z^4}, \\
B_2 &= \frac{\eta}{z^2} dx^+ \wedge ((\mu_1 \cos \psi_1) d(\mu_2 \cos \psi_2) - (\mu_2 \cos \psi_2) d(\mu_1 \cos \psi_1)) \\
&= \frac{\eta}{z^2} dx^+ \wedge (n_1 dn_3 - n_3 dn_1).
\end{aligned} \tag{3.20}$$

3) classical r -matrix with n_{15}

The third example is a classical r -matrix,

$$r = -\frac{1}{2} p_- \wedge n_{15}. \tag{3.21}$$

The resulting metric and NS-NS two-form are given by

$$\begin{aligned}
ds^2 &= ds_{\text{AdS}_5}^2 + ds_{\text{S}^5}^2 - \eta^2(\mu_1^2 \cos^2 \psi_1 + \mu_3^2 \sin^2 \psi_3) \frac{(dx^+)^2}{z^4} \\
&= ds_{\text{AdS}_5}^2 + ds_{\text{S}^5}^2 - \eta^2(n_1^2 + n_6^2) \frac{(dx^+)^2}{z^4}, \\
B_2 &= \frac{\eta}{z^2} dx^+ \wedge ((\mu_1 \cos \psi_1) d(\mu_3 \sin \psi_3) - (\mu_3 \sin \psi_3) d(\mu_1 \cos \psi_1)) \\
&= \frac{\eta}{z^2} dx^+ \wedge (n_1 dn_6 - n_6 dn_1).
\end{aligned} \tag{3.22}$$

4) classical r -matrix with n_{12} , n_{23} and n_{34}

The fourth example is a classical r -matrix,

$$r = \frac{1}{4} p_- \wedge (\alpha_1 n_{12} + \alpha_2 n_{23} + \alpha_3 n_{34}). \quad (3.23)$$

The resulting metric and NS-NS two-form are given by

$$\begin{aligned} ds^2 &= ds_{\text{AdS}_5}^2 + ds_{\text{S}^5}^2 \\ &\quad - \eta^2 (\alpha_1^2 (n_1^2 + n_2^2) + \alpha_2^2 (n_2^2 + n_3^2) + \alpha_3^2 (n_3^2 + n_4^2) - 2\alpha_2 (\alpha_1 n_1 n_3 + \alpha_3 n_2 n_4)) \frac{(dx^+)^2}{z^4}, \\ B_2 &= \frac{\eta}{z^2} dx^+ \wedge (\alpha_1 (n_1 dn_2 - n_2 dn_1) + \alpha_2 (n_2 dn_3 - n_3 dn_2) + \alpha_3 (n_3 dn_4 - n_4 dn_3)). \end{aligned} \quad (3.24)$$

5) classical r -matrix with n_{12} , n_{34} and h_6

The fifth example is a classical r -matrix,

$$r = \frac{1}{4} p_- \wedge \left(\alpha_1 n_{12} + \alpha_2 n_{34} - \frac{i\alpha_3}{2} h_6 \right). \quad (3.25)$$

Here it is noted that three $\mathfrak{so}(6)$ generators n_{12} , n_{34} and h_6 commute each other. The resulting metric and NS-NS two-form are given by

$$\begin{aligned} ds^2 &= ds_{\text{AdS}_5}^2 + ds_{\text{S}^5}^2 - \eta^2 (\alpha_1^2 (n_1^2 + n_2^2) + \alpha_2^2 (n_3^2 + n_4^2) + \alpha_3^2 (n_5^2 + n_6^2)) \frac{(dx^+)^2}{z^4}, \\ B_2 &= \frac{\eta}{z^2} dx^+ \wedge (\alpha_1 (n_1 dn_2 - n_2 dn_1) \\ &\quad + \alpha_2 (n_3 dn_4 - n_4 dn_3) + \alpha_3 (n_5 dn_6 - n_6 dn_5)). \end{aligned} \quad (3.26)$$

It should be remarked that the background with (3.26) agrees with the one found in [32]. Thus the existence of the associated classical r -matrix (3.25) indicates that the string theory defined on the background [32] is classically integrable in the sense that the Lax pair is constructed.

According to the Killing spinor analysis in [32], for values satisfying the condition

$$\alpha_1 \pm \alpha_2 \pm \alpha_3 = 0, \quad (3.27)$$

two real supersymmetries are preserved. When, in addition to the condition (3.27), at least one of the α_i ($i = 1, 2, 3$) vanishes, four real supersymmetries are preserved. Thus, depending on classical r -matrices, the remaining supersymmetries should be different. Hence it would be interesting to study the relation between classical r -matrices and the classification of super Schrödinger algebras [38]. It would also be nice to study the relation to warped AdS₃ geometries, for example, along the line of [39].

4 Integrability of gravity duals for dipole theories

In this section, we shall derive gravity duals of dipole theories [40–43] from the viewpoint of the Yang-Baxter deformations. We find out classical r -matrices associated with the dipole backgrounds and derive the deformed metric and NS-NS two-form both for a one-parameter case and a three-parameter case, in subsection 4.1 and 4.2, respectively.

4.1 A one-parameter case

Let us first consider a one-parameter dipole background. It can be obtained by a TsT-transformation $(x^3, \psi_1)_{\alpha_1}$, where α_1 is a shift parameter α_1 [44]. From this information on the TsT-transformation, one can easily guess the corresponding classical r -matrix based on the knowledge obtained [26].

A natural candidate of the associated r -matrix is given by

$$r = \frac{i\alpha_1}{4\eta} p_3 \wedge h_4, \quad (4.1)$$

where p_3 is a Poincaré generator in $\mathfrak{so}(2, 4)$ and h_4 is a Cartan generator in $\mathfrak{so}(6)$. In fact, the classical r -matrix leads to the deformed metric and NS-NS two form given by, respectively,

$$\begin{aligned} ds^2 &= \frac{-(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + G_1^{-1}(dx^3)^2 + dz^2}{z^2} \\ &\quad + d\mu_1^2 + G_1^{-1}\mu_1^2 d\psi_1^2 + \sum_{i=2,3} (d\mu_i^2 + \mu_i^2 d\psi_i^2), \\ B_2 &= \alpha_1 G_1^{-1} z^{-2} \mu_1^2 dx^3 \wedge d\psi_1. \end{aligned} \quad (4.2)$$

Here the scalar function G_1 is defined as

$$G_1 \equiv 1 + \alpha_1^2 \mu_1^2 z^{-2}. \quad (4.3)$$

We have also used the coordinates of S^5 given in (3.12). Indeed, the background with (4.2) perfectly agrees with (6.30) in [44]. Thus it has been shown that the classical r -matrix (4.1) corresponds to this TsT transformation. This result gives a further support for the gravity/CYBE correspondence.

4.2 A three-parameter case

The next is to consider a three-parameter generation of the classical r -matrix (4.2).

Since $\mathfrak{so}(6)$ has the three Cartan generators h_4, h_5 and h_6 , one may consider the following three-parameter generalization;

$$r = \frac{i}{4\eta} p_3 \wedge (\alpha_1 h_4 + \alpha_2 h_5 + \alpha_3 h_6), \quad (4.4)$$

with three real constants α_1, α_2 and α_3 . Note that, when $\alpha_2 = \alpha_3 = 0$, it reduces to the one-parameter case (4.2). After some computations, the resulting metric and NS-NS two-form are given by

$$\begin{aligned} ds^2 &= \frac{-(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + G_3^{-1}(dx^3)^2 + dz^2}{z^2} \\ &\quad + \sum_{i=1}^3 (d\mu_i^2 + \mu_i^2 d\psi_i^2) - G_3^{-1} z^{-2} \left(\sum_{i=1}^3 \alpha_i \mu_i^2 d\psi_i \right)^2, \\ B_2 &= z^{-2} G_3^{-1} dx^3 \wedge \left(\sum_{i=1}^3 \alpha_i \mu_i^2 d\psi_i \right). \end{aligned} \quad (4.5)$$

Here the scalar function G_3 is defined as

$$G_3 \equiv 1 + z^{-2} (\alpha_1^2 \mu_1^2 + \alpha_2^2 \mu_2^2 + \alpha_3^2 \mu_3^2). \quad (4.6)$$

When $\alpha_1 = \alpha_2 = \alpha_3 = 0$, G_3 becomes 1 and B_2 vanishes.

The deformed background with (4.5) can also be obtained by a sequence of three TsT-transformations: 1) $(x^3, \psi_1)_{\alpha_1}$, 2) $(x^3, \psi_2)_{\alpha_2}$ and 3) $(x^3, \psi_3)_{\alpha_3}$. The derivation is straightforward but the result has not been listed in [44]. Hence we have derived the explicit expressions in Appendix C. The resulting metric and NS-NS two-form completely agree with the expressions in (4.5).

5 Conclusion and Discussion

We have presented further examples of the gravity/CYBE correspondence. In the previous works, classical r -matrices have been composed of generators of only one of either $\mathfrak{so}(2, 4)$ or $\mathfrak{so}(6)$. In this paper, we consider some examples of r -matrices with both of them. The r -matrices of this kind contain (generalized) Schrödinger spacetimes and gravity duals of dipole field theories. It is known that the generalized Schrödinger spacetimes can also be obtained via a certain class of TsT transformations called null Melvin twists. The metric and NS-NS two-form are reproduced by following the Yang-Baxter sigma-model description. Thus, this agreement shows that these backgrounds are classically integrable at the bosonic

sigma-model level in the sense that the Lax pairs exist. We have to make efforts to prove this statement including the fermionic sector.

So far, the gravity/CYBE correspondence has been confirmed only for the metric (in the string frame) and NS-NS two-form. Hence the remaining task is to check the dilaton and R-R sector. In general, the dilaton is not constant and the R-R sector is also very complicated. Thus it does not seem so easy to check the remaining sector by explicitly evaluating the operator insertion into the classical action. However, the Schrödinger spacetime with a one-parameter is a bit special. Although the metric is deformed and the NS-NS two-form is newly turned on, the dilaton is still constant and the R-R sector is not modified. This result indicates that the Schrödinger spacetime would be a nice laboratory to check the gravity/CYBE correspondence at the full-sector level. We hope that we could report the result in the near future [45].

There are various applications of the results presented here. An exciting issue is to consider applications to integrable deformations of type IIA string theory on $\text{AdS}_4 \times \mathbb{CP}^3$ [46]. This system is dual to the $\mathcal{N}=6$ $SU(N) \times SU(N)$ Chern-Simons matter system in three dimensions [47]. This system was proposed by Aharony-Bergman-Jafferis-Maldacena (ABJM) [47] and it is often called the ABJM model. Hence the deformations of the string-theory side should correspond to deformations of the ABJM model, and there should be the associated classical r -matrices in the spirit of the gravity/CYBE correspondence.

In particular, the most significant one is a non-relativistic limit of the ABJM model [48]. This system preserves a super Schrödinger symmetry and the internal symmetry is also revealed. However, the gravity dual for this non-relativistic ABJM model has not been constructed yet⁷. Thus, it may be interesting to try to find out the gravity dual by employing the Yang-Baxter deformations of type IIA string theory. The information on the isometry obtained in [48] would be a key ingredient to find out the corresponding classical r -matrix. We hope that our results shed light on the gravity dual for the non-relativistic ABJM model.

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⁷Early trials to look for the gravity dual [49, 50] have supposed the five-dimensional Schrödinger geometries. This is a possible line of approach, but as another possibility one of the internal directions may be external by acting a classical r -matrix. This is what we have in mind.

Appendix

A Notation and convention

We summarize here the notation and convention of the $\mathfrak{so}(2, 4)$ and $\mathfrak{so}(6)$ generators, and a coset representation of $\text{AdS}_5 \times \text{S}^5$.

The gamma matrices

In the following, we use the gamma-matrices represented by

$$\begin{aligned} \gamma_1 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, & \gamma_2 &= \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, & \gamma_3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ \gamma_0 &= i\gamma_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & \gamma_5 &= i\gamma_1\gamma_2\gamma_3\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \end{aligned} \quad (\text{A.1})$$

To describe the $\mathfrak{so}(2, 4)$ and $\mathfrak{so}(6)$ subalgebras of the $\mathfrak{psu}(2, 2|4)$ superalgebra, it is necessary to introduce the following 8×8 gamma matrices:

$$\begin{aligned} \gamma_\mu^a &= \begin{pmatrix} \gamma_\mu & 0 \\ 0 & 0 \end{pmatrix}, & \gamma_5^a &= \begin{pmatrix} \gamma_5 & 0 \\ 0 & 0 \end{pmatrix} & \text{with} & \quad \mu = 1, 2, 3, 0, \\ \gamma_i^s &= \begin{pmatrix} 0 & 0 \\ 0 & \gamma_i \end{pmatrix}, & \gamma_5^s &= \begin{pmatrix} 0 & 0 \\ 0 & \gamma_5 \end{pmatrix} & \text{with} & \quad i = 1, 2, 3, 4. \end{aligned}$$

Here each block of the matrices is a 4×4 matrix.

The bosonic generators

Then, the Lie algebras $\mathfrak{so}(2, 4)$ and $\mathfrak{so}(6)$ are spanned by the bases:

$$\begin{aligned} \mathfrak{so}(2, 4) &= \text{span}_{\mathbb{R}} \{ \gamma_\mu^a, \gamma_5^a, m_{\mu\nu} = \frac{1}{4}[\gamma_\mu^a, \gamma_\nu^a], m_{\mu 5} = \frac{1}{4}[\gamma_\mu^a, \gamma_5^a] \mid \mu, \nu = 1, 2, 3, 0 \}, \\ \mathfrak{so}(6) &= \text{span}_{\mathbb{R}} \{ \gamma_i^s, \gamma_5^s, n_{ij} = \frac{1}{4}[\gamma_i^s, \gamma_j^s], n_{i5} = \frac{1}{4}[\gamma_i^s, \gamma_5^s] \mid i, j = 1, 2, 3, 4 \}. \end{aligned} \quad (\text{A.2})$$

Note that the subalgebras $\mathfrak{so}(1, 4)$ and $\mathfrak{so}(5)$ are generated by

$$\begin{aligned} \mathfrak{so}(1, 4) &= \text{span}_{\mathbb{R}} \{ m_{\mu\nu}, m_{\mu 5} \mid \mu, \nu = 1, 2, 3, 0 \}, \\ \mathfrak{so}(5) &= \text{span}_{\mathbb{R}} \{ n_{ij}, n_{i5} \mid i, j = 1, 2, 3, 4 \}. \end{aligned} \quad (\text{A.3})$$

For the coset construction of AdS_5 with the Poincaré coordinates, the following basis of $\mathfrak{so}(2, 4)$ is convenient;

$$\mathfrak{so}(2, 4) = \text{span}_{\mathbb{R}} \{ p_\mu, k_\mu, h_1, h_2, h_3, m_{13}, m_{10}, m_{23}, m_{20} \mid \mu = 0, 1, 2, 3 \}, \quad (\text{A.4})$$

where the Cartan generators h_1, h_2, h_3 and p_μ, k_μ are given by

$$\begin{aligned} h_1 &= 2im_{12} = \text{diag}(-1, 1, -1, 1, 0, 0, 0, 0), & p_\mu &= \frac{1}{2}\gamma_\mu^a - m_{\mu 5}, \\ h_2 &= 2m_{30} = \text{diag}(-1, 1, 1, -1, 0, 0, 0, 0), & k_\mu &= \frac{1}{2}\gamma_\mu^a + m_{\mu 5}, \\ h_3 &= \gamma_5^a = \text{diag}(1, 1, -1, -1, 0, 0, 0, 0). \end{aligned} \quad (\text{A.5})$$

Note that the generators p_μ and k_μ commute each other,

$$[p_\mu, p_\nu] = [k_\mu, k_\nu] = [p_\mu, k_\nu] = 0 \quad \text{for} \quad \mu, \nu = 0, 1, 2, 3. \quad (\text{A.6})$$

On the other hand, the Cartan generators of $\mathfrak{so}(6)$ read

$$\begin{aligned} h_4 &= 2in_{12} = \text{diag}(0, 0, 0, 0, -1, 1, -1, 1), \\ h_5 &= 2in_{34} = \text{diag}(0, 0, 0, 0, -1, 1, 1, -1), \\ h_6 &= \gamma_5^s = \text{diag}(0, 0, 0, 0, 1, 1, -1, -1). \end{aligned} \quad (\text{A.7})$$

A parameterization of the bosonic group elements

We are now ready to parametrize bosonic group elements of $PSU(2, 2|4)$. The group elements of $SO(2, 4)$ and $SO(6)$ are parametrized as

$$\begin{aligned} g_a &= \exp(x^1 p_1 + x^2 p_2 + x^3 p_3 + x^0 p_0) \exp\left(\frac{1}{2} \log z \gamma_5^a\right) \\ &= \exp(x^1 p_1 + x^2 p_2 + x^+ p_+ + x^- p_-) \exp\left(\frac{1}{2} \log z \gamma_5^a\right) \in SO(2, 4), \\ g_s &= \exp(\psi^1 h_4 + \psi^2 h_5 + \psi^3 h_6) \exp(-\zeta n_{13}) \exp\left(-\frac{i}{2} r \gamma_1^s\right) \in SO(6). \end{aligned}$$

Here the light-cone coordinates and the associated generators are given by

$$x^\pm = \frac{x^0 \pm x^3}{\sqrt{2}}, \quad p_\pm = \frac{p_0 \pm p_3}{\sqrt{2}}. \quad (\text{A.8})$$

Thus, a bosonic element g of $PSU(2, 2|4)$ is represented by

$$g = g_a g_s \in SO(2, 4) \times SO(6) \subset PSU(2, 2|4). \quad (\text{A.9})$$

Coset projector

To derive the metric of $\text{AdS}_5 \times \text{S}^5$ from the left-invariant one-form,

$$A = g^{-1} dg \in \mathfrak{so}(2, 4) \oplus \mathfrak{so}(6), \quad (\text{A.10})$$

it is necessary to introduce the coset projector;

$$P_2 : \quad \mathfrak{so}(2, 4) \oplus \mathfrak{so}(6) \quad \longrightarrow \quad \frac{\mathfrak{so}(2, 4)}{\mathfrak{so}(1, 4)} \oplus \frac{\mathfrak{so}(6)}{\mathfrak{so}(5)}. \quad (\text{A.11})$$

For any $x \in \mathfrak{so}(2, 4) \oplus \mathfrak{so}(6)$, it is explicitly defined as

$$P_2(x) = \frac{1}{4}(\eta^{\mu\nu}\gamma_\mu^a \text{Tr}[\gamma_\nu^a x] + \gamma_5^a \text{Tr}[\gamma_5^a x] + \delta^{ij}\gamma_i^s \text{Tr}[\gamma_j^s x]), \quad (\text{A.12})$$

where the range of the indices are $\mu, \nu = 0, 1, 2, 3$ and $i, j = 1, 2, 3, 4, 5$. Then the four-dimensional Minkowski metric is given by

$$\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1). \quad (\text{A.13})$$

The $\text{AdS}_5 \times \text{S}^5$ metric

With the left-invariant one-form (A.10) and the coset projector (A.12), the $\text{AdS}_5 \times \text{S}^5$ metric can be reproduced as

$$\text{STr}[AP_2(A)] = ds_{\text{AdS}_5}^2 + ds_{\text{S}^5}^2, \quad (\text{A.14})$$

where $ds_{\text{AdS}_5}^2$ and $ds_{\text{S}^5}^2$ are the metrics of AdS_5 and S^5 respectively,

$$\begin{aligned} ds_{\text{AdS}_5}^2 &= \frac{1}{z^2}(-2dx^+ dx^- + (dx^1)^2 + (dx^2)^2 + dz^2), \\ ds_{\text{S}^5}^2 &= dr^2 + \sin^2 r (d\zeta^2 + \cos^2 \zeta (d\psi_1)^2 + \sin^2 \zeta (d\psi_2)^2) + \cos^2 r (d\psi_3)^2. \end{aligned} \quad (\text{A.15})$$

This is the standard representation of the $\text{AdS}_5 \times \text{S}^5$ metric. It is noted that the ranges of S^5 coordinates are restricted as follows:

$$0 \leq r \leq \frac{\pi}{2}, \quad 0 \leq \zeta \leq \frac{\pi}{2}, \quad 0 \leq \psi_i \leq 2\pi \quad (i = 1, 2, 3). \quad (\text{A.16})$$

The coordinate systems of S^5

When we argue the Schrödinger geometry, it is more convenient to rewrite the S^5 metric (A.15) as an S^1 -fibration over \mathbb{CP}^2 :

$$\begin{aligned} ds_{\text{S}^5}^2 &= (d\chi + \omega)^2 + ds_{\mathbb{CP}^2}^2, \\ ds_{\mathbb{CP}^2}^2 &= d\mu^2 + \sin^2 \mu (\Sigma_1^2 + \Sigma_2^2 + \cos^2 \mu \Sigma_3^2), \end{aligned} \quad (\text{A.17})$$

where χ is the S^1 -fiber coordinate and ω is the one-form potential of the Kähler form on \mathbb{CP}^2 . The symbols Σ_i ($i = 1, 2, 3$) and ω are defined by

$$\Sigma_1 = \frac{1}{2}(\cos \psi d\theta + \sin \psi \sin \theta d\phi),$$

$$\begin{aligned}\Sigma_2 &= \frac{1}{2}(\sin \psi d\theta - \cos \psi \sin \theta d\phi), \\ \Sigma_3 &= \frac{1}{2}(d\psi + \cos \theta d\phi), \quad \omega = \sin^2 \mu \Sigma_3.\end{aligned}\tag{A.18}$$

The coordinate transformation from (A.15) to (A.17) is given by

$$\begin{aligned}\psi_1 &= \chi + \frac{1}{2}(\psi + \phi), & r &= \mu, \\ \psi_2 &= \chi + \frac{1}{2}(\psi - \phi), & \zeta &= \frac{1}{2}\theta, \\ \psi_3 &= \chi.\end{aligned}\tag{A.19}$$

B The T-duality rules

The rules of T-duality [51, 52] are summarized here. We basically follow the rules of [52].

The transformation rules between type IIB and type IIA supergravities are listed below. Note that the T-duality is performed for the y -direction and the other coordinates are denoted by a, b, a_i ($i = 1, \dots$). The fields of type IIB supergravity are the metric $g_{\mu\nu}$, NS-NS two-form B_2 , dilaton Φ , R-R gauge fields $C^{(2n)}$. The ones of type IIA supergravity are denoted with the tilde, the metric $\tilde{g}_{\mu\nu}$, NS-NS two-form \tilde{B}_2 , dilaton $\tilde{\Phi}$, and R-R gauge fields $\tilde{C}^{(2n+1)}$.

From type IIB to type IIA

$$\begin{aligned}\tilde{g}_{yy} &= \frac{1}{g_{yy}}, & \tilde{g}_{ay} &= \frac{B_{ay}}{g_{yy}}, & \tilde{g}_{ab} &= g_{ab} - \frac{g_{ya}g_{yb} - B_{ya}B_{yb}}{g_{yy}}, \\ \tilde{B}_{ay} &= \frac{g_{ay}}{g_{yy}}, & \tilde{B}_{ab} &= B_{ab} - \frac{g_{ya}B_{yb} - B_{ya}g_{yb}}{g_{yy}}, & \tilde{\Phi} &= \Phi - \frac{1}{2} \ln g_{yy}, \\ \tilde{C}_{a_1 \dots a_{2n+1}}^{(2n+1)} &= -C_{a_1 \dots a_{2n+1}y}^{(2n+2)} - (2n+1)B_{y[a_1}C_{a_2 \dots a_{2n+1}]y}^{(2n)} + 2n(2n+1)\frac{B_{y[a_1}g_{a_2|y]}C_{a_3 \dots a_{2n+1}]y}^{(2n)}}{g_{yy}}, \\ \tilde{C}_{a_1 \dots a_{2n}y}^{(2n+1)} &= C_{a_1 \dots a_{2n}}^{(2n)} + 2n\frac{g_{y[a_1}C_{a_2 \dots a_{2n}]y}^{(2n)}}{g_{yy}}.\end{aligned}\tag{B.1}$$

where the anti-symmetrization for indices is defined as, for example,

$$A_{[a}B_{b]} \equiv \frac{1}{2}(A_a B_b - A_b B_a).$$

The symbol $|y|$ inside the anti-symmetrization means that the indices other than the index y are anti-symmetrized.

From type IIA to type IIB

$$g_{yy} = \frac{1}{\tilde{g}_{yy}}, \quad g_{ay} = \frac{\tilde{B}_{ay}}{\tilde{g}_{yy}}, \quad g_{ab} = \tilde{g}_{ab} - \frac{\tilde{g}_{ya}\tilde{g}_{yb} - \tilde{B}_{ya}\tilde{B}_{yb}}{\tilde{g}_{yy}},$$

$$\begin{aligned}
B_{ay} &= \frac{\tilde{g}_{ay}}{\tilde{g}_{yy}}, & B_{ab} &= \tilde{B}_{ab} - \frac{\tilde{g}_{ya}\tilde{B}_{yb} - \tilde{B}_{ya}\tilde{g}_{yb}}{\tilde{g}_{yy}}, & \Phi &= \tilde{\Phi} - \frac{1}{2} \ln \tilde{g}_{yy}, \\
C_{a_1 \dots a_{2n}}^{(2n)} &= \tilde{C}_{a_1 \dots a_{2n}y}^{(2n+1)} - 2n\tilde{B}_{y[a_1}\tilde{C}_{a_2 \dots a_{2n}]y}^{(2n-1)} + 2n(2n-1)\frac{\tilde{B}_{y[a_1}\tilde{g}_{a_2|y]}\tilde{C}_{a_3 \dots a_{2n}]y}^{(2n-1)}}{\tilde{g}_{yy}}, \\
C_{a_1 \dots a_{2n-1}y}^{(2n)} &= -\tilde{C}_{a_1 \dots a_{2n-1}}^{(2n-1)} - (2n-1)\frac{\tilde{g}_{y[a_1}\tilde{C}_{a_2 \dots a_{2n-1}]y}^{(2n-1)}}{\tilde{g}_{yy}}.
\end{aligned} \tag{B.2}$$

C Derivation of dipole deformations of $\text{AdS}_5 \times \text{S}^5$

Let us here derive gravity duals for dipole theories by performing TsT transformations for $\text{AdS}_5 \times \text{S}^5$. In the following, we will concentrate on the NS-NS sector. The T-duality rules we utilize is summarized in Appendix B.

C.1 One-parameter deformation

First of all, we consider a one-parameter deformation of $\text{AdS}_5 \times \text{S}^5$. The starting point is the following metric, NS-NS two-form B_2 and dilaton Φ :

$$\begin{aligned}
ds^2 &= \frac{-(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + dz^2}{z^2} + \sum_{i=1}^3 (d\mu_i^2 + \mu_i^2 d\psi_i^2), \\
B_2 &= 0, \quad \Phi = \Phi_0 \text{ (const.)}.
\end{aligned} \tag{C.1}$$

For this background, we will perform a TsT transformation $(x^3, \psi_1)_{\alpha_1}$. For this purpose, the relevant part is

$$ds^2 = \frac{1}{z^2} (dx^3)^2 + \mu_1^2 d\psi_1^2, \quad B_2 = 0, \quad \Phi = \Phi_0. \tag{C.2}$$

The first step is a T-duality for the x^3 -direction. The resulting background is given by

$$d\tilde{s}^2 = z^2 (d\tilde{x}^3)^2 + \mu_1^2 d\psi_1^2, \quad B_2 = 0, \quad \Phi = \Phi_0 - \frac{1}{2} \ln \left(\frac{1}{z^2} \right). \tag{C.3}$$

Then, by shifting ψ_1 as $\psi_1 = \tilde{\psi}_1 - \alpha_1 \tilde{x}^3$, the background (C.3) is rewritten as

$$\begin{aligned}
d\tilde{s}^2 &= (z^2 + \mu_1^2 \alpha_1^2) (d\tilde{x}^3)^2 + \mu_1^2 d\tilde{\psi}_1^2 - 2\mu_1^2 \alpha_1 d\tilde{\psi}_1 d\tilde{x}^3, \\
\tilde{B}_2 &= 0, \quad \tilde{\Phi} = \Phi_0 + \frac{1}{2} \ln z^2.
\end{aligned} \tag{C.4}$$

Finally, a T-duality is performed for the \tilde{x}^3 -direction. Together with the undeformed part, the resulting background is given by

$$ds^2 = \frac{-(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + G_1^{-1} (dx^3)^2 + dz^2}{z^2} + \sum_{i=1}^3 (d\mu_i^2 + \mu_i^2 d\psi_i^2),$$

$$B_2 = \alpha_1 G_1^{-1} z^{-2} \mu_1^2 dx^3 \wedge d\psi_1, \quad e^{2\Phi} = e^{2\Phi_0} G_1^{-1}, \quad (\text{C.5})$$

Here we have removed the tilde from ψ_1 and a scalar function $G_1(z)$ is defined as

$$G_1(z) \equiv 1 + \alpha_1^2 \mu_1^2 z^{-2}.$$

C.2 Two-parameter deformation

We will next consider the second TsT transformation $(x^3, \psi_2)_{\alpha_2}$ for the background (C.5). The relevant part is

$$\begin{aligned} ds^2 &= G_1^{-1} \frac{(dx^3)^2}{z^2} + G_1^{-1} \mu_1^2 d\psi_1^2 + \mu_2^2 d\psi_2^2, \\ B_{\psi_1 x^3} &= -\frac{\mu_1^2 \alpha_1}{z^2 + \mu_1^2 \alpha_1}, \quad \Phi = \Phi_0 - \frac{1}{2} \ln G_1. \end{aligned} \quad (\text{C.6})$$

We first perform a T-duality for the x^3 -direction. The above part is rewritten as

$$\begin{aligned} d\tilde{s}^2 &= z^2 G_1 (d\tilde{x}^3)^2 + \mu_1^2 d\psi_1^2 + \mu_2^2 d\psi_2^2 - 2\mu_1^2 \alpha_1 d\psi_1 d\tilde{x}^3, \\ \tilde{B}_2 &= 0, \quad \tilde{\Phi} = \Phi_0 + \frac{1}{2} \ln z^2. \end{aligned} \quad (\text{C.7})$$

Then, by shifting ψ_2 as $\psi_2 = \tilde{\psi}_2 - \alpha_2 \tilde{x}^3$, the background (C.7) is rewritten as

$$\begin{aligned} d\tilde{s}^2 &= (z^2 G_1 + \alpha_2^2 \mu_2^2) (d\tilde{x}^3)^2 + \mu_1^2 d\psi_1^2 + \mu_2^2 d\tilde{\psi}_2^2 - 2\mu_1^2 \alpha_1 d\psi_1 d\tilde{x}^3 - 2\mu_2^2 \alpha_2 d\tilde{\psi}_2 d\tilde{x}^3, \\ \tilde{B}_2 &= 0, \quad \tilde{\Phi} = \Phi_0 + \frac{1}{2} \ln z^2. \end{aligned} \quad (\text{C.8})$$

Finally, we perform a T-duality for the \tilde{x}^3 -direction. Together with the undeformed part, the resulting background is given by

$$\begin{aligned} ds^2 &= \frac{-(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + G_2^{-1} (dx^3)^2 + dz^2}{z^2}, \\ &\quad + \sum_{i=1}^3 (d\mu_i^2 + \mu_i^2 d\psi_i^2) - z^{-2} G_2^{-1} (\alpha_1 \mu_1^2 d\psi_1 + \alpha_2 \mu_2^2 d\psi_2)^2, \\ B_2 &= z^{-2} G_2^{-1} dx^3 \wedge (\alpha_1 \mu_1^2 d\psi_1 + \alpha_2 \mu_2^2 d\psi_2), \quad e^{2\Phi} = e^{2\Phi_0} G_2^{-1}, \end{aligned} \quad (\text{C.9})$$

Here we have removed the tilde from ψ_2 and a scalar function $G_2(z)$ is defined as

$$G_2(z) \equiv 1 + z^{-2} (\alpha_1^2 \mu_1^2 + \alpha_2^2 \mu_2^2).$$

C.3 Three-parameter deformation

Finally, let us perform the third TsT transformation $(x^3, \psi_3)_{\alpha_3}$ for the background (C.9). The relevant part is given by

$$ds^2 = G_2^{-1} \frac{(dx^3)^2}{z^2} + \sum_{i=1}^3 \mu_i^2 d\psi_i^2 - z^{-2} G_2^{-1} (\alpha_1 \mu_1^2 d\psi_1 + \alpha_2 \mu_2^2 d\psi_2)^2, \\ B_{\psi_1 x^3} = -\frac{\mu_1^2 \alpha_1}{z^2} G_2^{-1}, \quad B_{\psi_2 x^3} = -\frac{\mu_2^2 \alpha_2}{z^2} G_2^{-1}, \quad \Phi = \Phi_0 - \frac{1}{2} \ln G_2. \quad (\text{C.10})$$

By performing a T-duality for the x^3 -direction, the above part is recast into

$$d\tilde{s}^2 = z^2 G_2 (d\tilde{x}^3)^2 + \sum_{i=1}^3 \mu_i^2 d\psi_i^2 - 2\alpha_1 \mu_1^2 d\psi_1 d\tilde{x}^3 - 2\alpha_2 \mu_2^2 d\psi_2 d\tilde{x}^3, \\ \tilde{B}_2 = 0, \quad \tilde{\Phi} = \Phi_0 + \frac{1}{2} \ln z^2. \quad (\text{C.11})$$

Then, by shifting ψ_3 as $\psi_3 = \tilde{\psi}_3 - \alpha_3 \tilde{x}^3$, the background (C.11) is rewritten as

$$d\tilde{s}^2 = (z^2 G_2 + \alpha_3^2 \mu_3^2) (d\tilde{x}^3)^2 + \mu_1^2 d\psi_1^2 + \mu_2^2 d\psi_2^2 + \mu_3^2 d\tilde{\psi}_3^2 \\ - 2\mu_1^2 \alpha_1 d\psi_1 d\tilde{x}^3 - 2\mu_2^2 \alpha_2 d\psi_2 d\tilde{x}^3 - 2\mu_3^2 \alpha_3 d\tilde{\psi}_3 d\tilde{x}^3, \\ \tilde{B}_2 = 0, \quad \tilde{\Phi} = \Phi_0 + \frac{1}{2} \ln z^2. \quad (\text{C.12})$$

Finally, we perform a T-duality for the \tilde{x}^3 -direction. Together with the undeformed part, the resulting background is given by

$$ds^2 = \frac{-(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + G_3^{-1} (dx^3)^2 + dz^2}{z^2}, \\ + \sum_{i=1}^3 (d\mu_i^2 + \mu_i^2 d\psi_i^2) - z^{-2} G_3^{-1} \left(\sum_{i=1}^3 \alpha_i \mu_i^2 d\psi_i \right)^2, \\ B_2 = z^{-2} G_3^{-1} dx^3 \wedge \left(\sum_{i=1}^3 \alpha_i \mu_i^2 d\psi_i \right), \quad e^{2\Phi} = e^{2\Phi_0} G_3^{-1}, \quad (\text{C.13})$$

Here we have removed the tilde from ψ_3 and a scalar function $G_3(z)$ is defined as

$$G_3(z) \equiv 1 + z^{-2} \sum_{i=1}^3 \alpha_i^2 \mu_i^2.$$

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